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RANDOM WALKS ON FUSION ALGEBRAS

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本講演は論文 [H] の内容に関することです.

A を II_1-factor, $\mathcal{C}$ を finite index A-A bimodule のなす tensor category とする (irreducible decomposition, direct sum, relative tensor product, unitary equivalence, conjugation で閉じているとする). $\mathcal{C}$ に属する irreducible bimodule の unitary equivalence class 全体を S とすると自由加群 $\mathbb{C}[S]$ は A-relative tensor product により積が定まり fusion algebra ([HI]) をなす. 本講演ではこのような bimodule のなす tensor category からできる fusion algebra 上での random walk について考察する. 記号の詳しい定義については [H] ([HY], [HI]) を参照して下さい.

Definition 1. Let $\mathbb{C}[S]$ be a fusion algebra with the multiplicative unit $I \in S$. Take two probability measures μ, ν on S and fix them. For a function $f \in l^\infty(S)$,

(1) f is *(μ, ν)-harmonic* if for $s \in S$,

$$f(s) = \sum_{t \in S} \mu * \delta_s * \nu(t) f(t).$$

(2) f is *left μ-harmonic* if it is (μ, δ_I)-harmonic.
(3) f is *right μ-harmonic* if it is (δ_I, μ)-harmonic.

The vector space consisting of all (μ, ν)-harmonic functions is a closed subspace of $l^\infty(S)$, which is referred to as the *(μ, ν)-harmonic function space.* Similarly we define left or right harmonic function spaces.

Notations.

(1) Let $\mathcal{C}$ be a C^*-tensor category and $\mathbb{C}[S]$ be the associated fusion algebra. For each $X, Y \in \mathrm{Object}(\mathcal{C})$, we write

$$XY = X \otimes Y,$$
$$\begin{bmatrix} X \\ Y \end{bmatrix} = \mathrm{Hom}(Y, X),$$
$$N_X^Y = \dim \begin{bmatrix} X \\ Y \end{bmatrix} = \dim \mathrm{Hom}(Y, X),$$
$$\delta_X = d(X)^{-1} \sum_{s \in S} N_X^s d(s) \delta_s.$$

(2) For a von Neumann algebra M, we denote its center by $Z(M)$.

S 上の probability measure μ に対し, [HY] で用いた構成法を使い, 各 bimodule X に対し von Neumann algebra $A_\infty(X)$ 及び bimodule X_∞ を作る. すると $Z(A_\infty(X))$ ($A_\infty(X)$ の center) と left μ-harmonic function が次の関係で一対一に対応する.

There exists a one to one correspondence between the left μ-harmonic function space and the center of $A_\infty(X)$ such that

$$E_{A_n(X)}(x) = \sum_{s \in S} f(s) I_{A_n^s(X)},$$

where $x \in Z(A_\infty(X))$ (the center of $A_\infty(X)$) and f is a left μ-harmonic function. ("E" means the trace-preserving conditional expectation.)

したがって left μ-harmonic function space を調べるには $Z(A_\infty)$ の構造を見ればよい.

まず, 次のことがわかる.

Proposition 2. *Let X be an object in $\mathcal{C}$. If XX^* generates the category $\mathcal{C}$, then the inclusion $A_\infty \subset A_\infty(X)$ is connected, i.e.,*

$$Z(A_\infty) \cap Z(A_\infty(X)) = \mathbb{C}.$$

この命題と inclusion $A_\infty \subset A_\infty(X)$ の Pimsner-Popa index が有限であることから次がでる. 証明は Jones tower

$$N \subset M \subset M_1 \subset \cdots \subset M_\infty$$

に対し $Z(N' \cap M_\infty)$ が atomic 又は diffuse であることを示すのと全く同じ方法でできる.

Proposition 3. *Assume that the tensor category $\mathcal{C}$ is finitely generated. Then the left μ-harmonic function space is either atomic or diffuse.*

[HY] の commuting square 構成法を ふたつの probability measure に拡張することで次がいえる.

Proposition 4. *The (μ, μ)-harmonic function space consists of only constant functions.*

このことから 特に次がいえる.

Corollary 5. *If $\mathbb{C}[S]$ is commutative, μ is always ergodic.*

したがって $\mathbb{C}[S]$ が可換の時, 自動的に weakly amenable となる ([HI][G] 参照).

次も Proposition 4 の簡単な系である. この系は Theorem 7 の証明で必要になる.

Corollary 6. *For any left μ-harmonic functions f and g, we have*

$$\lim_{k \to \infty} \lim_{n \to \infty} \sum_{s,t \in S} f(s) g(t) \mu^n(s) \delta_s * \mu^k(t) = f(I) g(I).$$

最後に fusion algebra が amenable の時を考察する. S. Popa は [P] において次のことを証明した.

Theorem (Popa [P]). *Let $N \subset M$ be an amenable inclusion of II_1-factors with finite index. Then almost ergodicity implies ergodicity.*

$N \subset M \subset M_1 \subset M_2 \subset \cdots \subset M_\infty$ を Jones tower とする．上の定理は $N \subset M$ が amenable ならば $Z(N' \cap M_\infty)$ は一次元又は無限次元のいずれかであることを主張している．この定理は一般の probability measure μ に拡張することができる．次の定理が主要結果である．

Theorem 7. *Assume that $\mathcal{C}$ is amenable. Let μ be a symmetric generating probability measure on S such that $I \in \mathrm{support}(\mu)$. Then the left μ-harmonic function space is either trivial or infinite dimensional.*

証明はやはり [HY] の構成法を利用する．我々の証明は subfactor の場合に制限しても Popa のそれとは全く異なり，別証明になっている．

以下この定理の証明の概略を述べる．$\mathcal{C}$ ($\mathbb{C}[S]$) が amenable であるとし $\dim Z(A_\infty) < \infty$ ならば $Z(A_\infty) = \mathbb{C}$ を示せばよい．$Z(A_\infty)$ の minimal projection による I の partition を $\{p_i\}_{i=1}^{n_0}$ とし 各 p_i に対応する left μ-harmonic function を f_i とおく．すると $\tau(p_i) = f_i(I)$ が成立している．

Definition 8. For each $X \in \mathrm{Object}(\mathcal{C})$, we define $n_0 \times n_0$ matrices as follows:

$$M(X)(i,j) = d({}_{A_\infty p_i} p_i X_\infty p_j {}_{A_\infty p_j}),$$
$$\Delta(X)(i,j) = [{}_{A_\infty p_i} p_i X_\infty p_j {}_{A_\infty p_j}]^{\frac{1}{2}},$$
$$L(X)(i,j) = \dim {}_{A_\infty p_i} p_i X_\infty p_j,$$
$$R(X)(i,j) = \dim p_i X_\infty p_j {}_{A_\infty p_j},$$

where $d(\cdot)$ denotes the quantum dimension (the square root of the minimal index), $[\ \cdot\ ]$ means Jones index, and "dim" is the coupling constant. Here we recall that the Jones index $[X]$ of an A-B bimodule X is given by

$$[X] = (\dim_A X) \cdot (\dim X_B).$$

$L(X)$ は次の形をしている．

Lemma 9. *For each $X \in \mathrm{Object}(\mathcal{C})$,*

$$L(X)(i,j) = \frac{d(X)}{\tau(p_i)} \lim_{n \to \infty} \sum_{s,t \in S} f_j(s) f_i(t) \mu^n(s) \delta_s * \delta_{X^*}(t).$$

matrix $M(X)$, $\Delta(X)$, $L(X)$, $R(X)$ はその定義から次の性質をみたすことがただちにわかる．

Lemma 10. *For each $X, Y \in \mathrm{Object}(\mathcal{C})$, the following statements hold:*

(1) $M(X \oplus Y) = M(X) + M(Y)$.
(2) $M(XY) = M(X)M(Y)$.
(3) $M(X^*) = {}^tM(X)$.
(4) $\Delta(XY)(i,j) \geq (\Delta(X)\Delta(Y))(i,j)$.
(5) $\Delta(X)(i,j) \geq M(X)(i,j)$.
(6) $\Delta(X)(i,j) = (L(X)(i,j)R(X)(i,j))^{\frac{1}{2}}$.
(7) $R(X)(i,j) = L(X^*)(j,i)$.

また Perron-Frobenius の定理を用いると次がいえる．

Lemma 11. *For each $X \in \mathrm{Object}(\mathcal{C})$ such that XX^* is a generator, the following inequality holds:*

$$||L_{XX^*}|| \leq ||M(XX^*)|| \leq ||\Delta(XX^*)|| \leq d(X)^2$$

where L_{XX^*} is the left regular representation of XX^* on $l^2(S)$ (see the definition of amenability of fusion algebras in [HY, Definition 2.1]), i.e.,

$$L_{XX^*}\delta_s = \sum_{t \in S} N_{XX^*s}^t \delta_t.$$

いま amenability を仮定しているので $||L_{XX^*}|| = d(X)^2$ が成立している．このことと上の補題から次が従う．

Corollary 12. *For each $X \in \mathrm{Object}(\mathcal{C})$, we have*

$$\Delta(X) = M(X).$$

したがって

$$d({}_{A_\infty p_i} p_i X_\infty p_j{}_{A_\infty p_j}) = [{}_{A_\infty p_i} p_i X_\infty p_j{}_{A_\infty p_j}]^{\frac{1}{2}}$$

となる．これは inclusion $A_\infty p_i \subset \mathrm{End}(p_i X_\infty p_j{}_{A_\infty p_j})$ が extremal になることを意味し次が成立する．

Lemma 13. *The number*

$$\alpha_{i,j} = \frac{R(X)(i,j)}{L(X)(i,j)}$$

does not depend on the choice of X whenever $L(X)(i,j) \neq 0$ and satisfies

$$\alpha_{i,j} \cdot \alpha_{j,k} = \alpha_{i,k}, \ \alpha_{i,j}^{-1} = \alpha_{j,i}.$$

一方 再び $||L_{XX^*}|| = d(X)^2$ を用いると次がわかる．

Lemma 14.

(1) *The family of matrices $\{\Delta(X)\}_{X \in \mathrm{Object}(\mathcal{C})}$ has a common Perron-Frobenius eigenvector*

$$\gamma = (\gamma_1, \cdots, \gamma_{n_0})$$

$(\gamma_i > 0)$ such that

$$\Delta(X)\gamma = d(X)\gamma.$$

(2) *For each $X \in \mathrm{Object}(\mathcal{C})$,*

$$||\Delta(X)|| = d(X).$$

Theorem 7 の証明. We have only to prove that, if the left μ-harmonic function space is finite dimensional, it is one-dimensional. We continue to use the notations as above. Recall that $p_i \in Z(A_\infty)$ corresponds to a left μ-harmonic function f_i. For each probability measure ν on S, define an $n_0 \times n_0$ matrix L_ν by

$$L_\nu(i,j) = \sum_{s \in S} \frac{\nu(s)}{d(s)} L(s)(i,j).$$

Although the notation is not explicit, we remark that $L(s)(i,j)$ depends on μ. By Lemma 9, we get

$$\begin{aligned} L_\nu(i,j) &= \frac{1}{\tau(p_i)} \lim_{n\to\infty} \sum_{s,t,u \in S} \nu(s) f_j(t) f_i(u) \mu^n(t) \delta_t * \delta_s(u) \\ &= \frac{1}{\tau(p_i)} \lim_{n\to\infty} \sum_{t,u \in S} f_j(t) f_i(u) \mu^n(t) \delta_t * \nu(u). \end{aligned}$$

This expression, together with the relation $L_{\nu^k} = (L_\nu)^k$, enables us to show that

$$\begin{aligned} L_\mu^k(i,j) &= L_{\mu^k}(i,j) \\ &= \frac{1}{\tau(p_i)} \lim_{n\to\infty} \sum_{t,u \in S} f_j(t) f_i(u) \mu^n(t) \delta_t * \mu^k(u) \\ &\to \tau(p_j) \end{aligned}$$

as $k \to \infty$. (Here we use Corollary 6.) For each probability measure ν on S, define

$$\Delta_\nu(i,j) = \sum_{s \in S} \frac{\nu(s)}{d(s)} \Delta(s)(i,j).$$

Then we compute

$$\begin{aligned} \Delta_\mu^k(i,j) &= \sum_{s \in S} \frac{\mu^k(s)}{d(s)} \Delta(s)(i,j) \\ &= \sum_{s \in S} \frac{\mu^k(s)}{d(s)} \alpha_{i,j}^{\frac{1}{2}} L(s)(i,j) = \alpha_{i,j}^{\frac{1}{2}} L_\mu^k(i,j). \end{aligned}$$

Thus $\Delta_\mu^k(i,j)$ tends to $\alpha_{i,j}^{\frac{1}{2}} \tau(p_j)$ as $k \to \infty$. On the other hand, by the definition of $\alpha_{i,j}$, we have

$$\alpha_{i,j} = \lim_{k\to\infty} \frac{L_\mu^k(j,i)}{L_\mu^k(i,j)} = \frac{\tau(p_i)}{\tau(p_j)}$$

and hence

$$\Delta_\mu^k(i,j) \to (\tau(p_i)\tau(p_j))^{\frac{1}{2}}.$$

This implies that the common Perron-Frobenius eigenvector γ obtained in Lemma 14 is proportional to $(\tau(p_1)^{\frac{1}{2}}, \cdots, \tau(p_{n_0})^{\frac{1}{2}})$. Therefore, for each $s \in S$,

$$\begin{aligned}
d(s)\tau(p_i)^{\frac{1}{2}} &= \sum_j \Delta(s)(i,j)\tau(p_j)^{\frac{1}{2}} \\
&= \sum_j \alpha_{i,j}^{\frac{1}{2}} L(s)(i,j)\tau(p_j)^{\frac{1}{2}} \\
&= \sum_j (\frac{\tau(p_i)}{\tau(p_j)})^{\frac{1}{2}} \{\frac{d(s)}{\tau(p_i)} \lim_{n\to\infty} \sum_{t,u\in S} f_j(t) f_i(u) \mu^n(t)\delta_t * \delta_{s^*}(u)\}\tau(p_j)^{\frac{1}{2}} \\
&= d(s)\frac{f_i(s^*)}{\tau(p_i)^{\frac{1}{2}}},
\end{aligned}$$

and we have $\tau(p_i) = f_i(s^*)$. This is equivalent to the relation $f_i(I) = f_i(s^*)$. Since s is arbitrary, each f_i must be a constant function, showing that μ is ergodic. □

Remark.

(1) 群の場合, left μ-harmonic function space は amenability とは無関係に常に trivial 又は diffuse となる (これは [K] で証明されている. 我々の方法で別証を与えることもできる. [H] 参照). しかし一般の bimodule からできる fusion algebra の場合, 2 次元になる例が U. Haagerup により得られている ([Ha]). もちろんこの時は non-amenable である.

(2) left μ-harmonic function space (特に $Z(N' \cap M_\infty)$) が無限次元かつ atomic であるような例があるかは (筆者の知るかぎり) まだわかっていない.

(3) fusion algebra の中には bimodule からこないタイプのものも存在する. このような fusion algebra にはいままでの議論は使えない.

References

[G] M. O. Gebuhrer, *Remarks on amenability of discrete hypergroups*, in "Harmonic Analysis and Discrete Potential Theory", ed. M. A. Picardello, Plenum Press, New York, 1992, pp. 131–143.

[Ha] U. Haagerup, talk at coferences "Oprrator Algebras and Applications", Geneva, 1994.

[H] T. Hayashi, *Harmonic function spaces of probability measures on fusion algebras*, preprint.

[HY] T. Hayashi and S. Yamagami, *Amenable tensor category and their realizations as AFD bimodules*, to appear in Jour. Funct. Anal.

[HI] F. Hiai and M. Izumi, *Amenability and strong amenability for fusion algebras with applications to subfactor theory*, Internat. Jour. Math. **9** (1998), 669–722.

[K] V. Kaimanovich, *The Poisson boundary of covering Markov operators*, Israel J. Math. **89** (1995), 77–134.

[P] S. Popa, *Some properties of the symmetric enveloping algebra of a subfactor, with applications to amenability and property T*, preprint.

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